Improvements of quasi-metrization theorems of Ribeiro and Pareek

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Axioms of set theory

I assume my modification ZF[W] of the axioms NBG of J. von Neumann (1903-1957), P. Bernays (1888-1977) and K. Gödel (1906-1978) that have their roots in Zermelo (1871-1953)-Fränkel (1891-1965) set theory ZF. The system ZF[W] includes neither the axioms of foundation nor replacement scheme, nor the axiom of choice. As usual ω stands for the class of all natural numbers of Zermelo-von Neumann. Let us denote by $[\omega]$ the statement:

the class ω is a set of elements.

This statement is unprovable in ZF[W]. Therefore, I assume ZF[W]+$[\omega]$ when the hypothesis of infinity is not deleted in my considerations.

Quasi-pseudometrics

A quasi-pseudometric d on a set X is a function $d : X \times X \to [0; +\infty)$ such that, for all x, y, z in X, the following conditions hold:

(i) $d(x, x) = 0$;

(ii) $d(x, z) \leqslant d(x, y) + d(y, z)$.

If, in addition, d satisfies the following

(iii) $d(x, y) = 0 \Rightarrow x = y$,

for all $x, y \in X$, then d is called a quasi-metric on X.

A non-Archimedean quasi-pseudometric on X is a function $d : X \times X \to [0; +\infty)$ such that, for all x, y, z in X, d satisfies (i) and the following strong triangle inequality:

(iv) $d(x, z) \leqslant \max\{d(x, y), d(y, z)\}$.

The topology $\tau(d)$ induced by a quasi-pseudometric d on X has the collection of all d-balls

$$B_d(x, \frac{1}{2^n}) = \{y \in X : d(x, y) < \frac{1}{2^n}\},$$

with $n \in \omega$, as a base of neighbourhoods at $x \in X$ for all points x of X.

Let $G = (G, \cdot)$ be a group with its identity e. Mimicking J. Marin and S. Romaguera (1996), an absolute quasi-value on G is a function $v : G \to [0; +\infty)$ such that:

(i) $v(e) = 0$;

(ii) $\forall_{x,y \in G} v(x \cdot y) \leqslant v(x) + v(y)$;

(iii) $\forall_{\epsilon>0} \forall_{x \in G} \exists_{\delta>0} \forall_{t \in G} (v(t) < \delta \Rightarrow v(x^{-1}tx) < \epsilon)$.

An absolute quasi-value v on G is non-Archimedean iff, for all $x, y \in G$, it satisfies the following:

$$v(x \cdot y) \leqslant \max\{v(x), v(y)\}.$$

If τ is a topology on G such that $\cdot : G \times G \to G$ is continuous with respect to τ, then $(G, \cdot, \tau)$ is called a quasi-topological group.
If v is an absolute quasi-value on G and, for all $x, y \in G$, we define

$$d_v(x, y) = v(x^{-1} \cdot y)$$

then d_v is a quasi-pseudometric on G such that $(G, \cdot, \tau(d_v))$ is a quasi-topological group.
Conversely, if d is a quasi-pseudometric on G such that $(G, \cdot, \tau(d))$ is a quasi-topological group and, for all $a, x, y \in G$, we have $d(ax, ay) = d(x, y)$ (i.e. d is left invariant), then the function v_d defined by $v_d(x) = d(e, x)$ for $x \in G$ is an absolute quasi-value on G and $d_{v_d} = d$.
Of course, v_d is non-Archimedean if and only if d is non-Archimedean.

A bitopological space is a triple (X, τ_0, τ_1) where X is a set and τ_o, τ_1 are topologies on X. Let d be a quasi-pseudometric on X. The conjugate d^{-1} is defined by

$$d^{-1}(x, y) = d(y, x) \text{ for } x, y \in X.$$

Then $(X, \tau(d), \tau(d^{-1}))$ is the bitopological space induced by d. Let $(G, \cdot, \tau)$ be a quasi-topological group. The conjugate τ^{-1} of τ is the collection of all sets $U \subseteq G$ such that $\{x^{-1} : x \in U\} \in \tau$. Then (G, τ, τ^{-1}) is the bitopological space associated with the quasi-topological group $(G, \cdot, \tau)$. Let d be a left invariant quasi-pseudometric on G such that $\tau = \tau(d)$. Then $\tau^{-1} = \tau(d^{-1})$. If v is an absolute quasi-value on G, then $(G, \tau(d_v), \tau({d_v}^{-1}))$ is the bitopological space associated with v.

A bitopological space (X, τ_0, τ_1) is called (non-Archimedeanly) quasi-(pseudo) metrizable iff there exists a (non-Archimedean) quasi-(pseudo)metric d on X such that $\tau_0 = \tau(d)$ and $\tau_1 = \tau(d^{-1})$.

Theorem

(Wajch) It is impossible to decide in ZF[W] whether there exist quasi-pseudometrizable bitopological spaces or they do not exist at all.

Assume that (X, τ_0, τ_1) is a bitopological space and assume that, for each $x \in X$, we are given a collection

$$G(x) = \{g(n, x) : n \in \omega\}$$

of subsets of X. Consider the following conditions:

(A) $\forall_{x,y \in X} \forall_{n \in \omega} (y \in g(n+1, x) \Rightarrow g(n+1, y) \subseteq g(n, x))$;

(B) $\forall_{x,y \in X} \forall_{n \in \omega} (y \in g(n, x) \Rightarrow g(n, y) \subseteq g(n, x))$;

(C) $\forall_{x,y \in X} \forall_{n \in \omega} (y \in g(n, x) \Leftrightarrow x \in g(n, y))$;

(D) for any $x \in X$, the collection $G(x)$ is a τ_0-base of neighbourhoods of x;

(E) for any $x \in X$, the collection $\{\{y \in X : x \in g(n, y)\} : n \in \omega\}$ is a τ_1-base of neighbourhoods of x.

Theorem

(Ribeiro 1943) In ZF[W]+[ω], the topological space (X, τ_0) is quasi-pseudometrizable if and only if there exists a collection $\{G(x) : x \in X\}$ such that $G(x) = \{g(n, x) : n \in \omega\}$ for each $x \in X$ and, moreover, conditions (A) and (D) are satisfied.

An improved version of Pareek's theorem (1979):

Theorem

In ZF[W]+[ω], the bitopological space (X, τ_0, τ_1) is quasi-pseudometrizable if and only if there exists a collection $\{G(x) : x \in X\}$ such that $G(x) = \{g(n, x) : n \in \omega\}$ for each $x \in X$ and, moreover, conditions (A),(D) and (E) are satisfied.

Theorem

(Fletcher, Lindgren, Nyikos, Gruenhage.) In ZF[W]+[ω], the topological space (X, τ_0) is non-Archimedeanly quasi-pseudometrizable if and only if there exists a collection $\{G(x) : x \in X\}$ such that $G(x) = \{g(n, x) : n \in \omega\}$ for each $x \in X$ and, moreover, conditions (B) and (D) are satisfied.

Theorem

(Wajch 2009) In ZF[W]+[ω], the bitopological space (X, τ_0, τ_1) is non-Archimedeanly quasi-pseudometrizable if and only if there exists a collection $\{G(x) : x \in X\}$ such that $G(x) = \{g(n, x) : n \in \omega\}$ for each $x \in X$ and, moreover, conditions (B), (D) and (E) are satisfied.

Theorem

(Wajch 2009) In ZF[W]+[ω], the topological space (X, τ_0) *is pseudometrizable if and only if there exists a collection* $\{G(x) : x \in X\}$ *such that* $G(x) = \{g(n, x) : n \in \omega\}$ *for each* $x \in X$ *and, moreover, conditions (A), (C) and (D) are satisfied.*

Theorem

(Wajch 2009) In ZF[W]+[ω], the bitopological space (X, τ_0, τ_1) *is non-Archimedeanly pseudometrizable if and only if there exists a collection* $\{G(x) : x \in X\}$ *such that* $G(x) = \{g(n, x) : n \in \omega\}$ *for each* $x \in X$ *and, moreover, conditions (B), (C) and (D) are satisfied.*

Theorem

(Marin, Romaguera 1994) In ZF[W]+[ω], the bitopological space associated with a quasi-topological group is quasi-pseudometrizable by a left invariant quasi-pseudometric if and only if the topology of the group is first-countable.

Theorem

(Wajch 2010) Let $(X, \cdot, \tau)$ be a quasi-topological group. In ZF[W]+[ω], the bitopological space (X, τ, τ^{-1}) associated with this group is quasi-pseudometrizable by a non-Archimedean left invariant quasi-pseudometric if and only if there exists a τ-base $\{U_n : n \in \omega\}$ of neighbourhoods of the identity e of this group such that $U_n \cdot U_n \subseteq U_n$ for each $n \in \omega$.

We use the ideas of G. Gruenhage (1977, 1984).
If d is a quasi-pseudometric on X such that $\tau_0 = \tau(d)$ and $\tau_1 = \tau(d^{-1})$, we put $g(n,x) = B_d(x, \frac{1}{2^n})$ for $x \in X$ and $n \in \omega$.

Bitopological spaces without algebraic structure.
To begin, assume that (X, τ_o, τ_1) is a bitopological space and, for each $x \in X$, the collection $\{g(n,x) : n \in \omega\}$ is a τ_0-base of neighbourhoods of x. For $x, y \in X$, let

$$A(x,y) = \{n \in \omega : y \notin g(n,x)\}.$$

We define a function $d : X \times X \to [0; +\infty)$ by putting

$$d(x,y) = 0 \text{ when } A(x,y) = \emptyset,$$

$$\text{whereas } d(x,y) = \frac{1}{2^{\min A(x,y)}} \text{ when } A(x,y) \neq \emptyset.$$

Suppose that condition (A) is satisfied. Then, for each positive real number ϵ, and for all $x, y, z \in X$, the following implication holds:

$$(d(x,y) < \epsilon \wedge d(y,z) < \epsilon) \Rightarrow d(x,z) < 2\epsilon.$$

By a result of Frink (1937), the function $\rho : X \times X \to [0; +\infty)$ defined by

$$\rho(x,y) = \inf\{\sum\nolimits_{i \in n} d(x_i, x_{i+1}) : n \in \omega \setminus \{0\}, \\ x_0 = x, x_n = y \wedge \forall_{i \in n} x_i \in X\}$$

is a quasi-pseudometric and has the following property:

$$(\star)\ \frac{1}{4} d(x,y) \leqslant \rho(x,y) \leqslant d(x,y)$$

for all $x, y \in X$.

Assume that condition (D) is also satisfied. To prove that $\tau_0 = \tau(\rho)$, it suffices to use $(\star)$ and check that

$$\forall_{x \in X} \forall_{n \in \omega} B_d(x, \frac{1}{2^n}) = g(n, x).$$

Since $B_{d^{-1}}(x, \frac{1}{2^n}) = \{y \in X : x \in g(n, y)\}$ for $x \in X$ and $n \in \omega$, if, in addition, condition (E) is fulfilled, then $\tau_1 = \tau(\rho^{-1})$.

Now, assume that conditions (B), (D) and (E) are all fulfilled. For $n \in \omega$ and $x \in X$, put $\tilde{g}(n,x) = \bigcap_{i \in n+1} g(i,x)$ and, similarly as d for g, define the function $\tilde{d}$ for $\tilde{g}$. Then $\tilde{d}$ is the required non-Archimedean quasi-pseudometric on X. If conditions (A), (C) and (D) are satisfied, ρ is a pseudometric inducing the topology τ_0. If conditions (B), (C) and (D) are fulfilled, $\tilde{d}$ is a non-Archimedean pseudometric such that $\tau_0 = \tau(\tilde{d})$.

Counter-example

Ribeiro and Pareek made a mistake when they claimed that d*, described by them in a more complicated way, satisfied the triangle inequality. For example, if* $X = \mathbb{R}$ *and* $g(n, x) = [x; x + \frac{1}{2^n})$ *for all* $x \in X$ *and* $n \in \omega$*, then* $d(0, \frac{1}{2}) = \frac{1}{2}, d(0, \frac{3}{8}) = \frac{1}{4}, d(\frac{3}{8}, \frac{1}{2}) = \frac{1}{8}$*, while* $d(0, \frac{1}{2}) > d(0, \frac{3}{8}) + d(\frac{3}{8}, \frac{1}{2})$*, so this* d *does not satisfy the triangle inequality.*

Counter-example

The non-Archimedean case of Gruenhage's proof in "Handbook of Set-Theoretic Topology" (10.2, p. 489) contains errors. Namely, we should not deduce from (B) that $g(n+1, x) \subseteq g(n, x)$ *or that* $B_d(x, \frac{1}{2^n}) = g(n, x)$*. For instance, if* $(r_n)_{n\in\omega}$ *is a sequence of all rational numbers such that* $r_0 = 1, r_1 = 3$ *and, for all* $n \in \omega$ *and* $x \in \mathbb{R}$*, we put* $g(n, x) = [x; r_n)$ *when* $x < r_n$*, while* $g(n, x) = [x; +\infty)$ *otherwise, then* $d(0, 2) = 1$ *and* $2 \in g(1, 0)$*, so* $B_d(0, \frac{1}{2}) \neq g(1, 0)$ *and* $g(1, 0)$ *is not a subset of* $g(0, 0)$*. Moreover, the function* d *in the proof of 10.2 of Chapter 10 of the book mentioned above is not well-defined because the largest* $n \in \omega$ *such that* $y \in g(n, x)$ *need not exist.*

First-countable quasi-topological groups.
Assume that $(X, \cdot, \tau)$ is a quasi-topological group such that there exists a τ-base $\{U_n : n \in \omega\}$ of neighbourhoods of the identity e of the group $(X, \cdot)$. Then there is a τ-base $\{V_n : n \in \omega\}$ of neighbourhoods of e such that $V_{n+1} \cdot V_{n+1} \subseteq V_n$ for each $n \in \omega$. The function ρ of 6.1 corresponding to $g(n, x) = x \cdot V_n$ for $x \in X$ and $n \in \omega$, is a left invariant quasi-pseudometric on X such that $\tau = \tau(\rho)$ and $\tau^{-1} = \tau(\rho^{-1})$. If $U_n \cdot U_n \subseteq U_n$ for each $n \in \omega$, then the function d of 6.1 corresponding to $g(n, x) = x \cdot \bigcap_{i \in n+1} U_i$ is a left invariant non-Archimedean quasi-pseudometric on X such that $\tau = \tau(d)$ and $\tau^{-1} = \tau(d^{-1})$.

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