CENTRAL SIMPLE ALGEBRAS, THE PROCESI-SCHACHER CONJECTURE, AND POSITIVE POLYNOMIALS

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Consider a central simple algebra A with involution $*$. The involution is called *positive* if the involution trace form $x \mapsto \operatorname{tr}(x^*x)$ is positive semidefinite (w.r.t. a fixed ordering of the center F of A). A symmetric element b is defined to be *positive* if the scaled involution trace form $x \mapsto \operatorname{tr}(x^*bx)$ is positive semidefinite, giving rise to an *ordering* of the central simple algebra A. We discuss how these can be used to give a Positivstellensatz characterizing polynomials in noncommuting variables that are positive semidefinite or trace-positive on $d \times d$ matrices. Along the way we give a counterexample to a conjecture of Procesi and Schacher. Here is a sample result:

Theorem *For a real polynomial f in n free noncommuting variables, the following are equivalent:*

(i) $\operatorname{tr}(f(A_1, \ldots, A_n)) \geq 0$ *for all* $A_i \in M_2(\mathbb{R})$*;*

(ii) there exist a nonvanishing central polynomial c, and a polynomial identity h of 2×2 matrices, such that

$$cfc^* \in h + \Theta^2.$$

Here Θ^2 denotes the set of all polynomials that can be written as sums of hermitian squares g^*g and commutators $pq - qp$.

The talk is partially based on joint work with Thomas Unger.