SINGULAR FUNCTIONS AND NORMAL NUMBERS

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A real number is called absolutely normal if for every $2 \le b \in \mathbb{N}$ in its base b expansion every block of digits occurs with the same frequency. A famous result of Borel (1909) is

Theorem 1 *Almost every real number is absolutely normal.*

This theorem can be proved in many ways. Some proofs use uniform distribution, combinatorics, probability or ergodic theory. There are also some elementary proofs of binary cases of Theorem 1.

We introduce the following class of functions. Let $\boldsymbol{b} = \{b_k\}_{k=1}^{\infty}$ be a sequence of integers $b_k \ge 2$. Let $\boldsymbol{\omega} = \{\omega_k\}_{k=1}^{\infty}$ be a sequence of divisions of the interval $[0,1]$,

$$\omega_k = \{\omega_k(c)\}_{c=0}^{b_k}, \quad \omega_k(0) = 0, \quad \omega_k(c) < \omega_k(c+1), \quad \omega_k(b_k) = 1.$$

Put

$$\Delta_k(c) := \omega_k(c+1) - \omega_k(c).$$

Function $\mathcal{F}_{\boldsymbol{b},\boldsymbol{\omega}} \colon [0,1] \to [0,1]$ corresponding to $\boldsymbol{b}$ and $\boldsymbol{\omega}$ is defined as follows. For $x \in [0,1)$, let

$$x = \sum_{n=1}^{\infty} \frac{c_n}{\prod_{k=1}^{n} b_k}$$

be its $\{b_k\}_{k=1}^{\infty}$-Cantor series. Then

$$\mathcal{F}_{\boldsymbol{b},\boldsymbol{\omega}}(x) := \sum_{n=1}^{\infty} \omega_n(c_n) \prod_{k=1}^{n-1} \Delta_k(c_k).$$

We define $\mathcal{F}_{\boldsymbol{b},\boldsymbol{\omega}}(1) = 1$.

We present another elementary proof of Theorem 1. The proof is based on basic properties of the above mentioned functions. We also use the facts that monotone function has finite derivative almost everywhere and that countable union of null sets is a null set.