AN ELEMENTARY CONSTRUCTIVE PROOF OF HILBERT'S THEOREM ON TERNARY QUARTICS

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I report on joint work with Claus Scheiderer (Konstanz). Hilbert's Theorem states that every positive semi-definite (psd) form F of degree 4 in 3 variables over the reals is a sum of 3 squares (of quadratic forms). We start from two results which were obtained several years ago by Powers-Reznick resp. by myself:

1. If F is positive definite (pd) then F is a sum of 3 squares iff a certain cubic equation E corresponding to F has a suitable integral solution
2. If F is semi-definite (sd) with a nontrivial real zero then F is always a sum of 3 squares

Our method is now as follows: For a given pd form F we construct a family F_t depending continuously on a real parameter t with $0 \leq t \leq 1$ and the corresponding equation E_t such that F_0 is sd and $F_1 = F$. For increasing value of t we show with the Implicit Function Theorem (IFT) that any admissible solution of E_t for fixed t can be extended uniquely to a solution of E'_t if $|t' - t|$ is small enough. Finally we get a solution of E_1 which gives the desired result. In order to apply IFT we have to exclude a finite number of rather explicite "special cases for F" where the full rank condition of IFT could fail. In these cases the proof is finished by an easy limit argument (already used by Hilbert).